LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

B.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIFTH SEMESTER – NOVEMBER 2007

MT 5502 - LINEAR ALGEBRA

AB 15

Date : 29/10/2007 Dept. No. Max. : 100 Marks
Time : 9:00 - 12:00

SECTION – A

Answer ALL the questions. (10 x 2 = 20 marks)

1. Let V be a vector space over F . If $a \in F, v \in V$ then prove that $(-a)v = a(-v) = -(av)$.
2. Prove that any subset of a linearly independent set is also linearly independent.
3. Prove that the vectors (1, 0, 0), (1, 1, 0) and (1, 1, 1) form a basis for R^3 .
4. Is the mapping $T : R^2 \to R$ defined by $T(a,b) = ab$ a homomorphism? Justify.
5. If V is an inner product space then prove that

$$< u, \alpha v + \beta w > = \overline{\alpha} < u, v > + \overline{\beta} < u, w >$$

6. Define eigenvalues and eigenvectors of a linear Transformation.
7. Define a unitary matrix and give an example.
8. If A and B are Hermitian show that $(AB + BA)$ is Hermitian and $(AB - BA)$ is skew Hermitian.
9. Find the rank of the matrix $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ over the field of rational numbers.
10. If $S, T \in A(V)$ then prove that $(ST)^* = T^* S^*$

SECTION – B

Answer any FIVE questions. (5 x 8 = 40 marks)

11. If S and T are subsets of a vector space V over F, then prove that
    i. S is a subset of V if and only if $L(S) = S$
    ii. $S \subseteq T$ implies that $L(S) \subseteq L(T)$
12. If V is a subspace of finite dimension and is the direct sum of its subspaces U and W, then prove that $\dim V = \dim U + \dim W$.
13. Let $T : U \to V$ be a homomorphism of two vector spaces over F, and suppose that U has finite dimension then prove that $\dim U = nullity\ of\ T + rank\ of\ T$
14. State and prove Schwarz inequality.
15. Prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is non zero.
16. If $V = R^3$ and let $T \in A(V)$ be defined by

$$T(a_1, a_2, a_3) = (3a_1 + a_3, -2a_1 + a_2, -a_1 + 2a_2 + 4a_3),$$

What is the matrix of T relative to the basis v_1 = (1, 0, 1), v_2 = (-1,2,1), v_3 = (2,1,1)

17. Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$ over the field of rational numbers.

18. Prove that if $<T(v), T(v)> = <v, v>$ for all v in V, T is Unitary.

SECTION – C

Answer any TWO questions. **(2 x 20 = 40 marks)**

19. (a) Prove that the vector space V over F is a direct sum of two of its subspaces W_1 and W_2 if and only if $V=W_1+W_2$ and $W_1 \cap W_2=(0)$.

(b) For $A, B \in F_n$ and $\lambda \in F$, Show that

(1) $tr(\lambda A) = \lambda tr(A)$

(2) $tr(A+B) = tr(A) + tr(B)$

(3) $tr(AB) = tr(BA)$

20. (a) Let V be a Vector space of finite dimension and W_1 and W_2 be subspaces of V such that $V = W_1 + W_2$ and $\dim V = \dim W_1 + \dim W_2$. Prove that $V = W_1 \oplus W_2$

(b) Let U and V be vector spaces over a field F. Prove that the set $Hom(U,V)$, the set of all homomorphisms from U to V is a vector space over F.

21. (a) Prove that every finite-dimensional inner product space V has an orthonormal basis.

(b) If $T \in A(V)$ and T satisfies a polynomial $f(x) \in F(x)$, prove that the minimal polynomial for T over F divides $f(x)$.

22. (a) Investigate for what values of λ, μ the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

over the rational field has (i) no solution, (ii) a unique solution (iii) an infinite number of solutions.

(b) Prove that the linear transformation T on V is unitary if and only if $T^*T=I$.

(c) Prove that the eigenvalues of a unitary transformation are all of absolute value 1.

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